

Causal Structure and Degenerate Phase Boundaries

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Abstract

Time-like and null hypersurfaces in the degenerate space-times in Ashtekar theory are defined in the light of the degenerate causal structure proposed by Matschull. Using the new definition of null hypersurfaces, the conjecture that the “phase boundary” separating the degenerate space-time region from the nondegenerate one in Ashtekar’s gravity is always null is proved under certain circumstances.

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I. INTRODUCTION

Ashtekar's formalism of general relativity[1] has led to a considerable progress in loop quantum gravity[2]. A special feature of this framework is that degenerate triads, and hence degenerate metrics, are admitted, and the degenerate metrics play an important role in the quantum description of gravity[3,4]. The significance of understanding degenerate metrics was emphasized in Refs.5 and 6. Various kinds of degenerate solutions to classical Ashtekar's equations have been studied[5-11], and the local causal structure of degenerate Ashtekar theory has also been established[12]. Using a "covariant approach", Bengtsson and Jacobson[6] investigated the structure of the "phase boundaries" between degenerate and nondegenerate space-time regions, and conjectured that the phase boundaries should always be null provided that the metric is a "regular" solution to Ashtekar's equations, that is, solutions in which the canonical variables (A_i^I, \tilde{e}_I^i) , the shift vector N^i , and the lapse density, \underline{N} (weight -1), all take finite value which, except for \underline{N} , are allowed to vanish. In a recent paper[13], however, a degenerate phase boundary is distinguished from its image, and moreover, it is shown that the definition of the nullness of the image of the phase boundary used in Ref.6 could not be generalized to the phase boundary itself. The main focus of the present paper, on the other hand, is first to give a reasonable definition of the nullness of the boundary, and then to prove the conjecture under certain circumstances.

The suggestion of Ref.6 to create a space-time with a degenerate region by the covariant approach is as follows. Start off with a nondegenerate metric which solves Einstein's equations and reparametrize one of the coordinates. This reparametrization is chosen so that it is not a diffeomorphism at some particular value of the coordinate. Adopting the new coordinate, the solution can be smoothly matched to a solution to the Ashtekar equations with a degenerate metric at the surface where the transformation misbehaves. To make things clearer we reformulate this procedure as follows. Let M be a 4-dimensional manifold and M_1 a 4-dimensional submanifold with a 3-dimensional boundary ∂M_1 . Suppose \hat{M} is a 4-dimensional manifold with a nondegenerate metric $\hat{g}_{\mu\nu}$ which solves the Einstein's equa-

tions, and ϕ is a diffeomorphism from M_1 to some open set $\hat{M}_1 \subset \hat{M}$. Extend the domain of ϕ smoothly to the whole of M so that $M - M_1$ is mapped onto $\phi[\partial M_1]$, and the pushforward ϕ_* restricted to the tangent bundle of ∂M_1 to that of $\phi[\partial M_1]$ is nondegenerate. Then the pullback $g_{\mu\nu} \equiv \phi^* \hat{g}_{\mu\nu}$ is nondegenerate on M_1 and degenerate on $M - M_1$. One therefore has a space-time $(M, g_{\mu\nu})$ with a “phase boundary”, ∂M_1 , separating a nondegenerate region from a degenerate one. It is clear that the “reparametrization procedure” mentioned above is a special case of this treatment.

Inspired by Ref.12, we try to define timelike and null hypersurfaces in a degenerate space-time in Sec.2. Armed with this new definition for a null hypersurface, we then give a proof of the conjecture that the phase boundary is null in Sec.3 under the circumstances where the degenerate space-time with a phase boundary is obtained by the reparametrization procedure mentioned above.

II. DEFINING NULL HYPERSURFACES IN DEGENERATE SPACE-TIMES

Suppose the boundary $\phi[\partial M_1]$ is given by $f = 0$, where f is a smooth function on \hat{M} with $\nabla_\mu f|_{\phi[\partial M_1]} \neq 0$, then $\phi[\partial M_1]$ is defined in Ref.[6] to be null if $g^{\mu\nu} \nabla_\mu f \nabla_\nu f \rightarrow 0$ as $\phi[\partial M_1]$ “is approached from the nondegenerate side”. However, as pointed out in Ref.[13], this definition is inappropriate to ∂M_1 since it depends upon the choice of the function f on M , and concrete examples show that there exist functions f and \bar{f} with $\lim g^{\mu\nu} \nabla_\mu f \nabla_\nu f = 0$ while $\lim g^{\mu\nu} \nabla_\mu \bar{f} \nabla_\nu \bar{f} \neq 0$. This obstacle could be overcome if we use $\sqrt{-g} g^{\mu\nu} \nabla_\mu f \nabla_\nu f$ instead of $g^{\mu\nu} \nabla_\mu f \nabla_\nu f$. In Ashtekar theory there is a well-defined densitized inverse metric, $\tilde{g}^{\mu\nu}$, with components in any coordinate system of a $3 + 1$ decomposition[12]:

$$\tilde{g}^{\mu\nu} = \begin{pmatrix} \tilde{g}^{tt} & \tilde{g}^{ti} \\ \tilde{g}^{jt} & \tilde{g}^{ji} \end{pmatrix} = \begin{pmatrix} -\underline{N}^{-1} & \underline{N}^{-1} N^i \\ \underline{N}^{-1} N^j & \underline{N} \tilde{h}^{ji} - \underline{N}^{-1} N^j N^i \end{pmatrix}, \quad (1)$$

where \underline{N} and N^i are respectively the lapse density and the shift vector, and \tilde{h}^{ji} is the densitized inverse 3-metric of weight +2, and in the nondegenerate case one has $\tilde{g}^{\mu\nu} =$

$\sqrt{-g}g^{\mu\nu}$. Eq.(1) implies that $\tilde{g}^{\mu\nu}$ remains finite in the Ashtekar theory of degenerate space-times, and therefore $\lim \sqrt{-g}g^{\mu\nu}\nabla_\mu f\nabla_\nu f = \tilde{g}^{\mu\nu}\nabla_\mu f\nabla_\nu f|_{f=0}$. Let f and \bar{f} be two distinct functions on M with $f|_{\partial M_1} = \bar{f}|_{\partial M_1} = 0$ and $\nabla_\mu f|_{f=0} \neq 0$ and $\nabla_\mu \bar{f}|_{\bar{f}=0} \neq 0$, then there exists a function λ on M such that $\nabla_\mu f|_{f=0} = \lambda \nabla_\mu \bar{f}|_{\bar{f}=0}$, and hence

$$\tilde{g}^{\mu\nu}\nabla_\mu f\nabla_\nu f|_{f=0} = \lambda^2 \tilde{g}^{\mu\nu}\nabla_\mu \bar{f}\nabla_\nu \bar{f}|_{\bar{f}=0}.$$

Hence, $\tilde{g}^{\mu\nu}\nabla_\mu f\nabla_\nu f|_{f=0} = 0$ if and only if $\tilde{g}^{\mu\nu}\nabla_\mu \bar{f}\nabla_\nu \bar{f}|_{\bar{f}=0} = 0$. We therefore obtain a self-consistent definition of null hypersurfaces in a degenerate space-time in Ashtekar's theory as follows:

Definition 1: A hypersurface described by $f = 0$ with $\nabla_\mu f|_{f=0} \neq 0$ in space-time $(M, g_{\mu\nu})$ is said to be null if $\tilde{g}^{\mu\nu}\nabla_\mu f\nabla_\nu f|_{f=0} = 0$.

In the following we will use the symbol, \tilde{E}_A^μ , to denote the vierbein of vector densities weighted $+1/2$, i.e., the square roots of $\tilde{g}^{\mu\nu}$, namely,

$$\tilde{g}^{\mu\nu} = \eta^{AB} \tilde{E}_A^\mu \tilde{E}_B^\nu,$$

where η^{AB} is the Minkowski metric to raise (and η_{AB} to lower) the interior indices “ A ” and “ B ”. Note that there is $SO(3,1)$ gauge freedom for \tilde{E}_A^μ , and the components of certain choice of \tilde{E}_A^μ in any coordinate system associated with a $3+1$ decomposition are

$$\tilde{E}_A^\mu = \begin{pmatrix} \tilde{E}_0^t & \tilde{E}_0^i \\ \tilde{E}_I^t & \tilde{E}_I^i \end{pmatrix} = \begin{pmatrix} \sqrt{N^{-1}} & -\sqrt{N^{-1}}N^i \\ 0 & -\sqrt{N}\tilde{e}_I^i \end{pmatrix}, \quad (2)$$

where \tilde{e}_I^i is the densitized triad of weight $+1$ in Ashtekar theory (with “ i ” and “ I ” the spatial and interior indices respectively), and the columns of (\tilde{E}_A^μ) are labelled by space-time indices. Given a vierbein, one may consider $\tilde{E}_A^\mu(x) : M^4 \rightarrow T_x M$ as a map from the 4-dimensional Minkowski space into the tangent bundle of the manifold M . The “future”, $\mathcal{F}(x)$, of a point, $x \in M$, can therefore be defined[12] as the set of all tangent vectors at x which are images of some vectors, ς^A , in M^4 satisfying $\varsigma^A \varsigma_A \leq 0$ and $\varsigma^0 > 0$, i.e.,

$$\mathcal{F}(x) \equiv \{v^\mu(x) \in T_x M \mid \exists \varsigma^A \in M^4, \varsigma^A \varsigma_A \leq 0, \varsigma^0 > 0, \text{ such that } v^\mu(x) = \varsigma^A \tilde{E}_A^\mu(x)\}.$$

Thus, depending on the rank of the vierbein, the future, $\mathcal{F}(x)$, is either a (4-dimensional) hypercone ($\text{rank } \tilde{E}_A^\mu = 4$), a (3-dimensional) cone ($\text{rank } \tilde{E}_A^\mu = 3$), an angle ($\text{rank } \tilde{E}_A^\mu = 2$), or a half-line ($\text{rank } \tilde{E}_A^\mu = 1$) in Ashtekar theory. This local causal structure can be used to define the timelike and null hypersurfaces as follows.

Definition 2: A hypersurface Σ is said to be timelike if for any point $x \in \Sigma$ the tangent space, $T_x\Sigma$ (tangent to Σ), of x contains a nonzero vector, v^μ , which is the image under the mapping \tilde{E}_A^μ of a timelike vector, ς^A , in the Minkowski space, i.e., $\exists v^\mu = \varsigma^A \tilde{E}_A^\mu \in T_x\Sigma$ such that $\eta_{AB}\varsigma^A\varsigma^B < 0$.

Definition 3: A hypersurface Σ is said to be null if for any point $x \in \Sigma$ the tangent space, $T_x\Sigma$, of x contains a nonzero vector that is the image of a null vector in the Minkowski space, and there exists no timelike vector, ς^A , in the Minkowski space such that $\varsigma^A \tilde{E}_A^\mu \in T_x\Sigma$.

Definitions 2 and 3 are consistent with the causal structure and can be re-formulated in terms of $\mathcal{F}(x)$ as follows: Let $i(\mathcal{F}(x))$ and $\partial\mathcal{F}(x)$ represent respectively the interior and the boundary of $\mathcal{F}(x)$, then a hypersurface Σ is timelike if and only if $T_x\Sigma \cap i(\mathcal{F}(x)) \neq \emptyset$, while Σ is null if and only if $T_x\Sigma \cap \mathcal{F}(x) \neq \emptyset$ and $T_x\Sigma \cap \mathcal{F}(x) \subset \partial\mathcal{F}(x)$. Note that the definition of a spacelike hypersurface Σ given by Ref.12 is equivalent to requiring $T_x\Sigma \cap \mathcal{F}(x) = \emptyset$. Note also that both Definitions 2 and 3 are applicable to the cases where the ranks of \tilde{E}_A^μ are two, three, and four. In the case where \tilde{E}_A^μ is of rank one, the timelike and null hypersurfaces become indistinguishable.

Now it is natural to ask whether Definition 3 is equivalent to Definition 1. Suppose a hypersurface defined by $f = 0$ with $\nabla_\mu f|_{f=0} \neq 0$ is null according to Definition 3, then any vector field, v^μ , tangent to the hypersurface satisfies

$$0 = v^\mu \nabla_\mu f = \varsigma^A \tilde{E}_A^\mu \nabla_\mu f = \varsigma^A \omega_A, \quad (3)$$

where ς^A is any inverse image of v^μ under \tilde{E}_A^μ , and $\omega_A \equiv \tilde{E}_A^\mu \nabla_\mu f$. Since ς^A can be null but not timelike, it follows from Eq.(3) that $\omega^B \equiv \eta^{BA}\omega_A$ must be a null vector. Consequently on $f = 0$ we have

$$\tilde{g}^{\mu\nu} \nabla_\mu f \nabla_\nu f = \eta^{AB} \tilde{E}_A^\mu \tilde{E}_B^\nu \nabla_\mu f \nabla_\nu f = \eta^{AB} \omega_A \omega_B = 0,$$

i.e., the hypersurface $f = 0$ is also null according to Definition 1. However, the degeneracy of \tilde{E}_A^μ implies the possibility of $\omega_A \equiv \tilde{E}_A^\mu \nabla_\mu f = 0$, in this case the hypersurface is null according to Definition 1 while might well be nonnull according to Definition 3.

The above arguments lead to the following equivalent definition of Definition 3:

Definition 3': A hypersurface $f = 0$ (with $\nabla_\mu f|_{f=0} \neq 0$) is null if $\omega_A \equiv \tilde{E}_A^\mu \nabla_\mu f|_{f=0}$ is a nonzero null covector in the Minkowski space.

Since Definition 3' is consistent with the local causal structure and convenient to use, we will use it to judge whether the phase boundary ∂M_1 is null in the next section.

It should be noted that the choice of the gauge as well as the coordinate system for \tilde{E}_A^μ is irrelevant. The interior gauge transformation preserves the Minkowski metric and hence does no harm to the previous discussions. Since \tilde{E}_A^μ are vector densities, for a vector ς^A in Minkowski space, the image $v^\mu(x) \equiv \tilde{E}_A^\mu \varsigma^A$, viewed as a vector at $x \in M$, will change under a coordinate transformation to $v'^\mu(x) \equiv \tilde{E}'^\mu_A \varsigma^A$. However, the transformation law for the components of a vector density guarantees that $v'^\mu(x)$ and $v^\mu(x)$ have the same direction, therefore coordinate transformations do no harm to the previous discussions either.

III. NULLNESS OF THE DEGENERATE PHASE BOUNDARY ∂M_1

We assume in this section that the degenerate phase boundary is obtained through the covariant approach mentioned in Sec.1. As shown in Ref.13, the hypersurface $\phi[\partial M_1]$ in \hat{M} must be null if the pullback metric $g_{\mu\nu}$ on M is to be a regular solution to Ashtekar's equations. The argument is as follows in short. In any 3+1 decomposition of the space-time $(M, g_{\mu\nu})$ one has $h_{ij}N^j = g_{0i}$ ($i = 1, 2, 3$), where h_{ij} and N^j are respectively the 3-metric and the shift vector. Since $h \equiv \det(h_{ij}) = 0$ in $M - M_1$, the last three columns of $(g_{\mu\nu})$ must be linearly dependent to ensure the finiteness of N^i . Hence there exists a 4-vector $T^\nu = (0, \lambda^i)$ at each point of $M - M_1$ with λ^i a non-vanishing 3-vector such that $g_{\mu\nu}T^\nu = 0$. Furthermore, the lapse scalar N must vanish to keep the lapse density \underline{N} finite in $M - M_1$, it then follows from $g_{00} = -N^2 + g_{0i}N^i$ that there exists another 4-vector $S^\nu = (1, -N^i)$ at

each point of $M - M_1$ such that $g_{\mu\nu}S^\nu = 0$. Therefore T^ν and S^ν represent two independent degenerate directions of $g_{\mu\nu}$, and hence there must be some degenerate vector field, W^ν , that is tangent to ∂M_1 . It follows from the nondegeneracy of the pushforward ϕ_* (restricted to ∂M_1) that there is a vector field, ϕ_*W^ν , on $\phi[\partial M_1]$ that is the desired null normal to $\phi[\partial M_1]$. It is also clear from this argument that the rank of $g_{\mu\nu}$ is two on ∂M_1 , from which one can argue that $\text{rank}(\tilde{g}^{\mu\nu}) = 2$, and hence $\text{rank}(\tilde{E}_A^\mu)$ must be two or three on ∂M_1 . We could therefore use Definition 3' in Sec.2 to judge whether ∂M_1 is null.

Without loss of generality, we choose a “time orthogonal” $3 + 1$ decomposition of the space-time $(\hat{M}, \hat{g}_{\mu\nu})$, and the line element reads

$$d\hat{s}^2 = -\hat{N}^2 dT^2 + \hat{h}_{ij} dX^i dX^j. \quad (4)$$

Let U be a smooth function on M with $\nabla_\mu U \neq 0$ and $U = 0$ represents $\phi[\partial M_1]$, then

$$dU = \hat{\beta}(T, X^j) dT + \hat{\alpha}_i(T, X^j) dX^i, \quad (5)$$

where $\hat{\beta} \equiv \partial U / \partial T$ and $\hat{\alpha}_i \equiv \partial U / \partial X^i$, $i = 1, 2, 3$. It follows from Eqs.(4), (5) and the nullness of $\phi[\partial M_1]$, i.e., $\hat{g}^{\mu\nu} \nabla_\mu U \nabla_\nu U|_{U=0} = 0$ that

$$[\hat{h}^{ij} \hat{\alpha}_i \hat{\alpha}_j - (\hat{\beta} / \hat{N})^2]|_{U=0} = 0, \quad (6)$$

where \hat{h}^{ij} is the inverse of the 3-metric \hat{h}_{ij} , and \hat{N} is the lapse scalar. The line element (4) in the domain of U can be re-expressed as

$$d\hat{s}^2 = (\hat{N} / \hat{\beta})^2 [-dU^2 + 2\hat{\alpha}_i dX^i dU - (\hat{\alpha}_i dX^i)^2] + \hat{h}_{ij} dX^i dX^j. \quad (7)$$

The mapping $\phi : M \rightarrow \hat{M}$ induces four functions $\phi^*U, \phi^*X^i (i = 1, 2, 3)$ on M with $\phi^*U|_{M-M_1} = 0$. Without essential loss of generality, let (u, x^i) be a local coordinate system on M covering a neighborhood of ∂M_1 with $u|_{\partial M_1} = 0, u|_{M_1} > 0$ [hereafter M_1 (or M) is short for “the interaction of M_1 (or M) and the coordinate patch”] and $x^i = \phi^*X^i$, then one has a function $U(u)$ [short for $(\phi^*U)(u)$] with $U'(u)|_{M-M_1} \equiv [dU/du]_{M-M_1} = 0$. It then follows from Eq.(7) that the line element of $g_{\mu\nu} \equiv \phi^*\hat{g}_{\mu\nu}$ in this coordinate system reads

$$ds^2 = (\hat{N}/\hat{\beta})^2 [-(U')^2 du^2 + 2U' \hat{\alpha}_i dx^i du - (\hat{\alpha}_i dx^i)^2] + \hat{h}_{ij} dx^i dx^j. \quad (8)$$

Let $u = u(t, x^i)$, where t is the time coordinate of certain $3 + 1$ decomposition of $(M, g_{\mu\nu})$, and

$$du = \beta(t, x^i) dt + \alpha_i(t, x^i) dx^i,$$

where $\beta \equiv \partial u / \partial t$ and $\alpha_i \equiv \partial u / \partial x^i$, $i = 1, 2, 3$, then the $3 + 1$ decomposition of the metric (8) reads

$$ds^2 = -(\hat{N}/\hat{\beta})^2 [(U'\beta)^2 dt^2 + 2U'\beta\gamma_i dx^i dt + (\gamma_i dx^i)^2] + \hat{h}_{ij} dx^i dx^j, \quad (9)$$

where $\gamma_i \equiv U'\alpha_i - \hat{\alpha}_i$. The determinant, g , of the line element (9), the spatial 3-metric, h_{ij} , induced by metric (9), and the determinant, h , of h_{ij} can be obtained through straightforward calculations as

$$g \equiv \det(g_{\mu\nu}) = -(U'\beta\hat{N}/\hat{\beta})^2 \hat{h}, \quad \hat{h} \equiv \det(\hat{h}_{ij}), \quad (10)$$

$$h_{ij} = \hat{h}_{ij} - (\hat{N}/\hat{\beta})^2 \gamma_i \gamma_j, \quad (11)$$

$$h \equiv \det(h_{ij}) = \hat{h}[1 - (\hat{N}/\hat{\beta})^2 \gamma_i \gamma_j \hat{h}^{ij}]. \quad (12)$$

Since $g = -N^2 h$, where N is the lapse scalar, it follows from Eqs.(10) and (12) that the Ashtekar's lapse density on M_1 is

$$\underline{N} = \frac{N}{\sqrt{h}} = \frac{\sqrt{-g}}{h} = \frac{|\beta\hat{\beta}U'|}{\hat{N}\sqrt{\hat{h}}[(\hat{\beta}/\hat{N})^2 - \gamma_i \gamma_j \hat{h}^{ij}]}. \quad (13)$$

Since the shift vector, N^i , relates to the metric components of Eq.(9) via

$$h_{ij} N^j = g_{0i} = -(\hat{N}/\hat{\beta})^2 \beta U' \gamma_i,$$

a straightforward calculation shows that the shift vector on M_1 reads

$$N^i = g_{0i} h^{ij} = \frac{U' \beta \gamma_j \hat{h}^{ij}}{\hat{h}^{lm} \gamma_l \gamma_m - (\hat{\beta}/\hat{N})^2}. \quad (14)$$

Using Eq.(6), it is not difficult to show that Eqs.(13) and (14) imply

$$\underline{N}|_{\partial M_1} = \lim_{u \rightarrow 0^+} \frac{|\beta \hat{\beta} U'|}{\hat{N} \sqrt{\hat{h}} [(\hat{\beta}/\hat{N})^2 - \gamma_i \gamma_j \hat{h}^{ij}]} = \frac{|\beta \hat{\beta}|}{\hat{N} \sqrt{\hat{h}} |2\hat{\alpha}_i \alpha_j \hat{h}^{ij} + B|}, \quad (15)$$

and

$$N^i|_{\partial M_1} = \lim_{u \rightarrow 0^+} \frac{U' \beta \gamma_j \hat{h}^{ij}}{\hat{h}^{lm} \gamma_l \gamma_m - (\hat{\beta}/\hat{N})^2} = \frac{\beta \hat{\alpha}_j \hat{h}^{ij}}{2\hat{h}^{lm} \hat{\alpha}_l \alpha_m + B}, \quad (16)$$

where

$$B \equiv \lim_{u \rightarrow 0^+} \frac{1}{U'} [(\hat{\beta}/\hat{N})^2 - \hat{h}^{ij} \hat{\alpha}_i \hat{\alpha}_j]. \quad (17)$$

If we choose the vierbein, \tilde{E}_A^μ , as Eq.(2), then, according to Definition 3', the key quantity for judging whether the hypersurface $u = \text{const.}$ is null is

$$u_A \equiv \tilde{E}_A^\mu (du)_\mu = \begin{pmatrix} \sqrt{\underline{N}^{-1}} (\beta - \alpha_i N^i) \\ -\sqrt{\underline{N}} \alpha_i \tilde{e}_I^i \end{pmatrix} \equiv \begin{pmatrix} u_0 \\ u_I \end{pmatrix}, \quad I = 1, 2, 3, \quad (18)$$

where

$$(du)_\mu = \begin{pmatrix} \beta \\ \alpha_i \end{pmatrix}, \quad i = 1, 2, 3.$$

It follows from Eqs.(15), (16), and (18) that

$$(u_0)^2|_{\partial M_1} = \left| \frac{\beta}{\hat{\beta}} \right| \frac{\hat{N} \sqrt{\hat{h}} (\hat{\alpha}_i \alpha_j \hat{h}^{ij} + B)^2}{|2\hat{h}^{lm} \hat{\alpha}_l \alpha_m + B|}, \quad (19)$$

and

$$(u_1)^2 + (u_2)^2 + (u_3)^2 = \underline{N} \alpha_i \alpha_j \tilde{e}_I^i \tilde{e}^{Ij} = \underline{N} \alpha_i \alpha_j \tilde{h}^{ij}. \quad (20)$$

Through a non-trivial calculation, which will be given in the Appendix, we get

$$\alpha_i \alpha_j \tilde{h}^{ij}|_{\partial M_1} = \hat{h} (\hat{N}/\hat{\beta})^2 (\hat{\alpha}_i \alpha_j \hat{h}^{ij})^2. \quad (21)$$

Hence Eq.(20) evaluated on ∂M_1 gives

$$[(u_1)^2 + (u_2)^2 + (u_3)^2]|_{\partial M_1} = \left| \frac{\beta}{\hat{\beta}} \right| \frac{\hat{N} \sqrt{\hat{h}} (\hat{\alpha}_i \alpha_j \hat{h}^{ij})^2}{2 \hat{h}^{lm} \hat{\alpha}_l \alpha_m + B}, \quad (22)$$

which equals Eq.(19) if and only if $B = 0$. To show that this condition is indeed satisfied we first obtain from Eq.(17) the following expression of B :

$$B = \lim_{u \rightarrow 0^+} \left[\frac{\partial(\hat{\beta}^2/\hat{N}^2 - \hat{h}^{ij} \hat{\alpha}_i \hat{\alpha}_j)}{\partial U} \frac{U'}{U''} \right], \quad (23)$$

where $U'' \equiv dU'/du$. If

$$b \equiv \lim_{u \rightarrow 0^+} \frac{U'}{U''} \neq 0, \quad (24)$$

there exists a smooth function $L(u)$ on M such that $U''/U' = L(u)$ on M_1 , and $\lim_{u \rightarrow 0^+} (U''/U') = L(0) = 1/b$. Therefore one has

$$U' = C \exp\left[\int_0^u L(\tau) d\tau\right] \quad \text{on } M_1 \quad (25)$$

where $C = \text{const.}$. Since $\lim_{u \rightarrow 0^+} U' = 0$, Eq.(25) implies that $C = 0$, and hence $U'|_{M_1} = 0$, contradicting the above mentioned requirement of the mapping ϕ . Therefore the assumption (24) is false, and we have

$$\lim_{u \rightarrow 0^+} \frac{U'}{U''} = 0,$$

and hence it follows from Eq.(23) that $B = 0$ since $\partial(\hat{\beta}^2/\hat{N}^2 - \hat{h}^{ij} \hat{\alpha}_i \hat{\alpha}_j)/\partial U$ is regular on M . Consequently the vector $u^A|_{\partial M_1}$ is null in the Minkowski space. Furthermore, since \underline{N} should be finite for a regular solution to Ashtekar's equations, Eq.(15) implies that Eq.(22) does not vanish, ensuring that $u^A|_{\partial M_1} \neq 0$. We therefore conclude that the phase boundary, ∂M_1 , represented by $u = 0$, is null.

It is also worth noting that, in contrast with what the authors of Ref.6 argued, (This is originally, in Ref.6, referred to the image rather than the phase boundary itself.) the Ashtekar's evolution equations are not at all a necessary condition for the degenerate phase boundary ∂M_1 to be null.

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APPENDIX A: PROOF FOR EQ.(21)

Denoting by A^{ij} and \hat{A}^{ij} the complementary minors of h_{ij} and \hat{h}_{ij} respectively, it follows from Eq.(11) that

$$\begin{aligned}
\alpha_i \alpha_j \tilde{h}^{ij} &= (-1)^{i+j} \alpha_i \alpha_j A^{ij} \\
&= (\alpha_1)^2 A^{11} + (\alpha_2)^2 A^{22} + (\alpha_3)^2 A^{33} + 2(-\alpha_1 \alpha_2 A^{12} + \alpha_1 \alpha_3 A^{13} - \alpha_2 \alpha_3 A^{23}) \\
&= (\alpha_1)^2 \{ \hat{A}^{11} - (\hat{N}/\hat{\beta})^2 [\hat{h}_{22}(\hat{\alpha}_3)^2 + \hat{h}_{33}(\hat{\alpha}_2)^2 - 2\hat{h}_{23}\hat{\alpha}_2\hat{\alpha}_3] \} \\
&\quad + (\alpha_2)^2 \{ \hat{A}^{22} - (\hat{N}/\hat{\beta})^2 [\hat{h}_{11}(\hat{\alpha}_3)^2 + \hat{h}_{33}(\hat{\alpha}_1)^2 - 2\hat{h}_{13}\hat{\alpha}_1\hat{\alpha}_3] \} \\
&\quad + (\alpha_3)^2 \{ \hat{A}^{33} - (\hat{N}/\hat{\beta})^2 [\hat{h}_{22}(\hat{\alpha}_1)^2 + \hat{h}_{11}(\hat{\alpha}_2)^2 - 2\hat{h}_{21}\hat{\alpha}_2\hat{\alpha}_1] \} \\
&\quad + 2\alpha_1 \alpha_2 \{ -\hat{A}^{12} + (\hat{N}/\hat{\beta})^2 [\hat{h}_{21}(\hat{\alpha}_3)^2 + \hat{h}_{33}\hat{\alpha}_1\hat{\alpha}_2 - \hat{h}_{13}\hat{\alpha}_2\hat{\alpha}_3 - \hat{h}_{23}\hat{\alpha}_1\hat{\alpha}_3] \} \\
&\quad + 2\alpha_3 \alpha_2 \{ -\hat{A}^{32} + (\hat{N}/\hat{\beta})^2 [\hat{h}_{23}(\hat{\alpha}_1)^2 + \hat{h}_{11}\hat{\alpha}_3\hat{\alpha}_2 - \hat{h}_{13}\hat{\alpha}_2\hat{\alpha}_1 - \hat{h}_{21}\hat{\alpha}_1\hat{\alpha}_3] \} \\
&\quad + 2\alpha_1 \alpha_3 \{ \hat{A}^{13} - (\hat{N}/\hat{\beta})^2 [\hat{h}_{21}\hat{\alpha}_2\hat{\alpha}_3 + \hat{h}_{32}\hat{\alpha}_1\hat{\alpha}_2 - \hat{h}_{13}(\hat{\alpha}_2)^2 - \hat{h}_{22}\hat{\alpha}_1\hat{\alpha}_3] \} \\
&= (-1)^{i+j} \alpha_i \alpha_j \hat{A}^{ij} - (\hat{N}/\hat{\beta})^2 [\hat{h}_{11}(\alpha_2 \hat{\alpha}_3 - \alpha_3 \hat{\alpha}_2)^2 + \hat{h}_{22}(\alpha_3 \hat{\alpha}_1 - \alpha_1 \hat{\alpha}_3)^2 \\
&\quad + \hat{h}_{33}(\alpha_2 \hat{\alpha}_1 - \alpha_1 \hat{\alpha}_2)^2 + 2\hat{h}_{21}(\alpha_3 \hat{\alpha}_2 - \alpha_2 \hat{\alpha}_3)(\alpha_1 \hat{\alpha}_3 - \alpha_3 \hat{\alpha}_1) \\
&\quad + 2\hat{h}_{32}(\alpha_1 \hat{\alpha}_2 - \alpha_2 \hat{\alpha}_1)(\alpha_3 \hat{\alpha}_1 - \alpha_1 \hat{\alpha}_3) + 2\hat{h}_{31}(\alpha_2 \hat{\alpha}_1 - \alpha_1 \hat{\alpha}_2)(\alpha_3 \hat{\alpha}_2 - \alpha_2 \hat{\alpha}_3)] \\
&= \hat{h} \hat{h}^{ij} \alpha_i \alpha_j - (\hat{N}/\hat{\beta})^2 \hat{h} \{ [\hat{h}^{22} \hat{h}^{33} - (\hat{h}^{23})^2] (\alpha_2 \hat{\alpha}_3 - \alpha_3 \hat{\alpha}_2)^2 \\
&\quad + [\hat{h}^{11} \hat{h}^{33} - (\hat{h}^{13})^2] (\alpha_3 \hat{\alpha}_1 - \alpha_1 \hat{\alpha}_3)^2 + [\hat{h}^{22} \hat{h}^{11} - (\hat{h}^{21})^2] (\alpha_2 \hat{\alpha}_1 - \alpha_1 \hat{\alpha}_2)^2 \\
&\quad + 2(\hat{h}^{31} \hat{h}^{23} - \hat{h}^{21} \hat{h}^{33}) (\alpha_3 \hat{\alpha}_2 - \alpha_2 \hat{\alpha}_3)(\alpha_1 \hat{\alpha}_3 - \alpha_3 \hat{\alpha}_1) \\
&\quad + 2(\hat{h}^{31} \hat{h}^{21} - \hat{h}^{11} \hat{h}^{32}) (\alpha_1 \hat{\alpha}_2 - \alpha_2 \hat{\alpha}_1)(\alpha_3 \hat{\alpha}_1 - \alpha_1 \hat{\alpha}_3) \}
\end{aligned}$$

$$+2(\hat{h}^{21}\hat{h}^{23} - \hat{h}^{31}\hat{h}^{22})(\alpha_3\hat{\alpha}_2 - \alpha_2\hat{\alpha}_3)(\alpha_2\hat{\alpha}_1 - \alpha_1\hat{\alpha}_2)\} \quad (\text{A1})$$

Let $H \equiv \hat{h}^{ij}\alpha_i\alpha_j$, then, from Eq.(6) one gets

$$(\hat{\beta}/\hat{N})^2 H|_{\partial M_1} = \hat{h}^{lm}\hat{\alpha}_l\hat{\alpha}_m\hat{h}^{ij}\alpha_i\alpha_j. \quad (\text{A2})$$

Denoting by L the last brace of Eq.(A1), it follows from Eqs.(A1) and (A2) that

$$\begin{aligned} \left(\frac{\hat{\beta}^2}{\hat{h}\hat{N}^2}\right)\alpha_i\alpha_j\tilde{h}^{ij}|_{\partial M_1} &= \hat{h}^{lm}\hat{\alpha}_l\hat{\alpha}_m\hat{h}^{ij}\alpha_i\alpha_j - L \\ &= (\hat{h}^{11}\alpha_1\hat{\alpha}_1)^2 + (\hat{h}^{22}\alpha_2\hat{\alpha}_2)^2 + (\hat{h}^{33}\alpha_3\hat{\alpha}_3)^2 + [\hat{h}^{12}(\alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1)]^2 \\ &\quad + [\hat{h}^{23}(\alpha_3\hat{\alpha}_2 + \alpha_2\hat{\alpha}_3)]^2 + [\hat{h}^{13}(\alpha_1\hat{\alpha}_3 + \alpha_3\hat{\alpha}_1)]^2 + 2[\hat{h}^{11}\hat{h}^{33}\alpha_1\alpha_3\hat{\alpha}_1\hat{\alpha}_3 \\ &\quad + \hat{h}^{11}\hat{h}^{22}\alpha_1\alpha_2\hat{\alpha}_1\hat{\alpha}_2 + \hat{h}^{22}\hat{h}^{33}\alpha_2\alpha_3\hat{\alpha}_2\hat{\alpha}_3 + \hat{h}^{12}\hat{h}^{23}(\alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1)(\alpha_3\hat{\alpha}_2 + \alpha_2\hat{\alpha}_3) \\ &\quad + \hat{h}^{12}\hat{h}^{13}(\alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1)(\alpha_3\hat{\alpha}_1 + \alpha_1\hat{\alpha}_3) + \hat{h}^{31}\hat{h}^{23}(\alpha_1\hat{\alpha}_3 + \alpha_3\hat{\alpha}_1)(\alpha_3\hat{\alpha}_2 + \alpha_2\hat{\alpha}_3)] \\ &\quad + 2(\hat{h}^{11}\alpha_1\hat{\alpha}_1 + \hat{h}^{22}\alpha_2\hat{\alpha}_2 + \hat{h}^{33}\alpha_3\hat{\alpha}_3)[\hat{h}^{12}(\alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1) + \hat{h}^{23}(\alpha_3\hat{\alpha}_2 + \alpha_2\hat{\alpha}_3) \\ &\quad + \hat{h}^{13}(\alpha_1\hat{\alpha}_3 + \alpha_3\hat{\alpha}_1)] \\ &= (\hat{h}^{ij}\alpha_i\hat{\alpha}_j)^2, \end{aligned}$$

and hence

$$\alpha_i\alpha_j\tilde{h}^{ij}|_{\partial M_1} = \hat{h}(\hat{N}/\hat{\beta})^2(\hat{\alpha}_i\alpha_j\hat{h}^{ij})^2.$$

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